Exercise 39

If $xy + e^y = e$, find the value of y'' at the point where $x = 0$.

Solution

Differentiate both sides with respect to x.

$$\frac{d}{dx}(xy + e^y) = \frac{d}{dx}(e)$$

$$\frac{d}{dx}(xy) + \frac{d}{dx}(e^y) = 0$$

$$\left[\frac{d}{dx}(x)\right] y + x\left[\frac{d}{dx}(y)\right] + \left[(e^y)\cdot\frac{d}{dx}(y)\right] = 0$$

$$(1)y + x(y') + (e^y \cdot y') = 0$$

$$y + xy' + y'e^y = 0$$

Solve for y'.

$$y + (x + e^y)y' = 0 \tag{1}$$

$$y' = -\frac{y}{x + e^y}$$

Differentiate both sides of equation (1) with respect to x to get y''.

$$\frac{d}{dx}[y + (x + e^y)y'] = \frac{d}{dx}(0)$$

$$\frac{d}{dx}(y) + \frac{d}{dx}[(x + e^y)y'] = 0$$

$$y' + \left[\frac{d}{dx}(x + e^y)\right] y' + (x + e^y)\left[\frac{d}{dx}(y')\right] = 0$$

$$y' + (1 + e^y \cdot y')y' + (x + e^y)(y'') = 0$$

Bring the terms with y' to the right side.

$$\begin{aligned}(x + e^y)y'' &= -2y' - (y')^2 e^y \\ &= -2\left(-\frac{y}{x + e^y}\right) - \left(-\frac{y}{x + e^y}\right)^2 e^y \\ &= \frac{2y}{x + e^y} - \frac{y^2}{(x + e^y)^2}e^y \\ &= \frac{2y(x + e^y) - y^2 e^y}{(x + e^y)^2}\end{aligned}$$

Now solve for y''.

$$\begin{aligned} y'' &= \frac{2y(x+e^y) - y^2e^y}{(x+e^y)^3} \\ &= \frac{2xy + 2ye^y - y^2e^y}{(x+e^y)^3} \end{aligned}$$

Plug in $x = 0$ to the given equation to find the corresponding y-value on the curve.

$$x = 0: \quad (0)y + e^y = e \quad \rightarrow \quad y = 1$$

Therefore, the value of y'' at the point where $x = 0$ is

$$y''(0,1) = \frac{2(0)(1) + 2(1)e^1 - (1)^2e^1}{(0+e^1)^3} = \frac{1}{e^2}.$$